Problem A.2

Use Equation A.13 to simplify the expression $\delta(\sqrt{x^2+1} - x - 1)$.

Solution

Equation A.13 is on page 426.

$$\delta(g(x)) = \sum_{i=1}^{n} \frac{1}{|g'(x_i)|}\delta(x - x_i) \tag{A.13}$$

In this problem $g(x) = \sqrt{x^2+1} - x - 1$. Find the zeros of this function.

$$\sqrt{x^2+1} - x - 1 = 0$$

$$\sqrt{x^2+1} = x + 1$$

$$x^2 + 1 = (x+1)^2$$

$$x^2 + 1 = x^2 + 2x + 1$$

$$0 = 2x$$

$$x = 0$$

Now evaluate the derivative of $g(x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}(\sqrt{x^2+1} - x - 1) \\ &= \frac{1}{2}(x^2+1)^{-1/2} \cdot \frac{d}{dx}(x^2+1) - 1 \\ &= \frac{1}{2\sqrt{x^2+1}} \cdot (2x) - 1 \\ &= \frac{x}{\sqrt{x^2+1}} - 1 \end{aligned}$$

Therefore, by Equation A.13,

$$\begin{aligned} \delta(\sqrt{x^2+1} - x - 1) &= \frac{1}{|g'(0)|}\delta(x - 0) \\ &= \frac{1}{\left|\frac{0}{\sqrt{0^2+1}} - 1\right|}\delta(x) \\ &= \delta(x). \end{aligned}$$